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CONFIDENCE INTERVALS ON A RATIO
OF VARIANCES IN THE TWO-FACTOR CROSSED
COMPONENTS OF VARIANCE MODEL

by

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In the model $Y_{ij} = \mu + A_i + T_j + \varepsilon_{ij}$ where A_i, T_j, ε_{ij} are independent normal random variables with zero means and variances σ_A^2, σ_T^2, and σ_ε^2 a method is presented and evaluated for setting approximate confidence intervals on $\sigma_A^2/(\sigma_A^2 + \sigma_T^2 + \sigma_\varepsilon^2)$ and $\sigma_T^2/(\sigma_A^2 + \sigma_T^2 + \sigma_\varepsilon^2)$.

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ABSTRACT

Consider the two-factor crossed components of variance model given by

$$Y_{ij} = \mu + A_i + T_j + \varepsilon_{ij} \quad \text{for} \quad i = 1, \ldots, I;\ j = 1, \ldots, J.$$

The random variables A_i, T_j, ε_{ij} are normal and independent with means zero and variances σ_A^2, σ_T^2, σ_ε^2. Approximate confidence intervals are presented and evaluated for the ratios $\rho_A = \sigma_A^2/(\sigma_A^2 + \sigma_T^2 + \sigma_\varepsilon^2)$ and $\rho_T = \sigma_T^2/(\sigma_A^2 + \sigma_T^2 + \sigma_\varepsilon^2)$.

1. Introduction.

Consider the two-factor crossed Components-of-Variance model given by

$$Y_{ij} = \mu + A_i + T_j + \epsilon_{ij} \qquad i=1,2,\ldots I;\ j=1,2,\ldots J\ .$$

The A_i, T_j, ϵ_{ij} are independent unobservable random variables and

$$A_i \sim N(0,\sigma_A^2) \qquad T_j \sim N(0,\sigma_T^2) \qquad \epsilon_{ij} \sim N(0,\sigma_\varepsilon^2)$$

μ is an unknown parameter, and the Y_{ij} are observable random variables.

The analysis of variance table for this model is given in Table 1.

Table 1.

Source	d.f.	S.S.	M.S.	E.M.S.
Total	I J	$\sum\sum Y_{ij}^2$		
Mean	1	$I\ J\ \bar{Y}^2$		
Due to A	$n_1 = (I-1)$	$\sum_i\sum_j (\bar{Y}_{i.}-\bar{Y}_{..})^2$	S_1^2	$\theta_1 = J\sigma_A^2 + \sigma_\varepsilon^2$
Due to T	$n_2 = (J-1)$	$\sum_i\sum_j (\bar{Y}_{.j}-\bar{Y}_{..})^2$	S_2^2	$\theta_2 = I\sigma_T^2 + \sigma_\varepsilon^2$
Error	$n_3 = n_1 n_2$	$\sum_i\sum_j (Y_{ij}-\bar{Y}_{i.}-\bar{Y}_{.j}+\bar{Y}_{..})^2$	S_3^2	$\theta_3 = \sigma_\varepsilon^2$

The random variables $\bar{Y}$, S_1^2, S_2^2, S_3^2 are complete sufficient statistics for this problem and $n_i S_i^2/\theta_i \sim \chi^2(n_i)$ for $i=1,2,3$ and are mutually independent.

An important problem in applied statistics is to obtain confidence intervals on various functions of the variance components σ_A^2, σ_T^2, and σ_ε^2.

For some well-known functions exact-size confidence intervals are available, but in many situations exact-size confidence intervals have not as yet been derived.

For the following important and useful functions exact-size confidence intervals are known (exact-size means exact confidence coefficient): σ_ε^2, $\sigma_\varepsilon^2/\sigma_A^2$, $\sigma_\varepsilon^2/\sigma_T^2$, $\sigma_\varepsilon^2/(\sigma_\varepsilon^2 + \sigma_A^2)$, $\sigma_\varepsilon^2/(\sigma_\varepsilon^2 + \sigma_T^2)$, $\sigma_A^2/(\sigma_\varepsilon^2 + \sigma_A^2)$, $\sigma_T^2/(\sigma_\varepsilon^2 + \sigma_T^2)$. For a discussion of these see Graybill (1976).

Very good approximate confidence intervals are known for σ_A^2 and σ_T^2

For approximate confidence intervals on linear combinations of σ_A^2, σ_T^2, σ_ε^2 see Welch (1956).

In applied work it is often important to determine the ratios of the individual variances σ_A^2, σ_T^2, and σ_ε^2 to the total variation $(\sigma_A^2 + \sigma_T^2 + \sigma_\varepsilon^2)$, i.e., to determine

$$\rho_A = \sigma_A^2/(\sigma_A^2 + \sigma_T^2 + \sigma_\varepsilon^2)$$

$$\rho_T = \sigma_T^2/(\sigma_A^2 + \sigma_T^2 + \sigma_\varepsilon^2)$$

$$\rho_\varepsilon = \sigma_\varepsilon^2/(\sigma_A^2 + \sigma_T^2 + \sigma_\varepsilon^2)$$

There is no method available for setting exact $1-\alpha$ confidence intervals on ρ_A , ρ_T , or ρ_ε. Large sample approximations have been proposed by Osborne and Paterson (1952).

The purpose of this article is to obtain good approximate $1-\alpha$ upper, lower, and two-sided confidence intervals for ρ_A and ρ_T and evaluate how close the approximate confidence coefficients are to the specified $1-\alpha$.

The quantities ρ_A and ρ_T are useful in many applications. Graybill (1976) gives an example of a two-factor crossed components of variance model without interaction. An investigator wants to examine the effects of drivers and automobiles on gas mileage. One type of automobile is used and 20 are selected at random from one month's production. Ten people are selected at random from a large city and are asked to drive each car for two weeks and record the number of miles per gallon of gas used for each car for the two-week period. The total variation in the number of miles per gallon of gas used for this type of automobile is $\sigma_A^2 + \sigma_T^2 + \sigma_\varepsilon^2$, where σ_A^2 is the variation due to the cars, σ_T is the variation due to the drivers, and σ_ε^2 is the variation due to all other noncontrolled factors. The proportion of the variation in the number of miles per gallon due to the cars is given by ρ_A , and the proportion of the variation in the number of miles per gallon due to the drivers is given by ρ_T.

2. Approximate Confidence Intervals.

In this section approximate confidence intervals on ρ_A and ρ_T will be derived.

First consider a $1-\alpha$ upper confidence interal on ρ_A denoted by $\ell_A \leq \rho_A$ where ℓ_A is the lower confidence point.

It is simpler to find a $1-\alpha$ upper confidence interval for

$$\theta_A = \frac{J\sigma_A^2}{I(\sigma_T^2 + \sigma_\varepsilon^2)} = \frac{\theta_1 - \theta_3}{\theta_2 + n_1\theta_3}$$

and convert this to a confidence interval for ρ_A by using the fact that

$$\rho_A = \left(\frac{I\theta_A}{J + I\theta_A}\right) \tag{2.1}$$

A function $g_L(\underline{Y})$ of the observable random vector $\underline{Y}$ must be determined such that

$$P[g_L(\underline{Y}) \leq \theta_A]$$

is close to a specified $1-\alpha$ for all values of the unknown parameters. Since $\overline{Y}$, S_1^2, S_2^2, S_3^2 is a set of complete sufficient statistics for this problem, the lower confidence point should be a function of these random variables. So a function $h_L(\overline{Y},S_1^2,S_2^2,S_3^2)$ must be determined such that

$$P[h_L(\overline{Y},S_1^2,S_2^2,S_3^2) \leq \theta_A]$$

is close to a specified $1-\alpha$.

If any constant k is added to each observation Y_{ij} the parameter θ_A is unchanged so it is required that $h_L(\overline{Y}+k,S_1^2,S_2^2,S_3^2)$

remains unchanged. The constant $k = -\overline{Y}$ is chosen and thus $h_L(\overline{Y}, S_1^2, S_2^2, S_3^2) = h_L(0, S_1^2, S_2^2, S_3^2) = q_L(S_1^2, S_2^2, S_3^2)$, which is a function of S_1^2, S_2^2, and S_3^2 only.

If each observation Y_{ij} is multiplied by a non-zero constant c, the parameter θ_A is unchanged so it is required that $q_L(S_1^2, S_2^2, S_3^2)$ remains unchanged also. The constant chosen is $1/S_3^2$ so

$$q_L(S_1^2, S_2^2, S_3^2) = q_L\left(\frac{S_1^2}{S_3^2}, \frac{S_2^2}{S_3^2}, 1\right) = L\left(\frac{S_1^2}{S_3^2}, \frac{S_2^2}{S_3^2}\right)$$

which is a function of $S_1^2/S_3^2 = F_1$ and $S_2^2/S_3^2 = F_2$ only.

The problem is reduced to determining a function $L(F_1, F_2)$ such that

$$P[L(F_1, F_2) \leq \theta_A]$$

will be close to a specified confidence coefficient $1-\alpha$.

To determine the function $L(F_1, F_2)$ the following five conditions are imposed.

1) The confidence interval must be exact when $\theta_3 = 0$.

When $\theta_3 = 0$ it follows that $S_3^2 = 0$ with probability one, $F_1^{-1} = 0$ with probability one, and $\theta_A = \theta_1/\theta_2$. So an exact $1-\alpha$ lower confidence limit on θ_A is

$$\frac{S_1^2}{F_{\alpha:n_1,n_2} S_2^2}$$

and $L(F_1, F_2)$ must be equal to

$$\frac{F_1}{F_{\alpha:n_1,n_2} F_2}$$

in order to satisfy this condition.

2) When $\sigma_A{}^2 \to \infty$ it is required that the confidence coefficient be exact. When $\sigma_A{}^2 \to \infty$ and $\sigma_T{}^2, \sigma_\varepsilon{}^2$ are fixed it follows that $\theta_1 \to \infty$ with θ_2, θ_3 fixed, and θ_1 dominates θ_A. So the confidence interval on θ_A should behave like an exact $1-\alpha$ confidence interval on θ_1. So $L(F_1, F_2)$ should be equal to $S_1{}^2 F^{-1}_{\alpha:n_1,\infty}$

3) The confidence interval is required to be exact when $J \to \infty$ and I is fixed.

When $J \to \infty$ it follows that $n_2 \to \infty$, $n_3 \to \infty$ and hence $S_2{}^2 \to \theta_2$ in probability and $S_3{}^2 \to \theta_3$ in probability; also,

$$(S_2{}^2 + n_1 S_3{}^2) \to (\theta_2 + n_1\theta_3)$$

in probability.

Begin with an exact $(1-\alpha)$ confidence interval for θ_1 given by

$$S_1{}^2 F^{-1}_{\alpha:n_1,\infty} \leq \theta_1$$

and subtract $S_3{}^2$ from the left side and the "equivalent" value θ_3

from the right side to obtain

$$S_1^2 F^{-1}_{\alpha:n_1,\infty} - S_3^2 \leq \theta_1 - \theta_3$$

Then divide the left and right sides, respectively, by $(S_2^2 + n_1 S_3^2)$ and the "equivalent" value $(\theta_2 + n_1\theta_3)$ to obtain

$$\frac{S_1^2 F^{-1}_{\alpha:n_1,\infty} - S_3^2}{S_2^2 + n_1 S_3^2} \leq \frac{\theta_1 - \theta_3}{\theta_2 + n_1\theta_3}$$

Hence, when $J \to \infty$ the lower confidence point should behave like

$$L(F_1,F_2) = \frac{F_1 F^{-1}_{\alpha:n_1,\infty} - 1}{F_2 + n_1}$$

4) When $I \to \infty$ and J is fixed the confidence interval is required to be exact.

For $I \to \infty$ and J fixed it follows that $n_1 \to \infty$, $n_3 \to \infty$ and $S_1^2 \to \theta_1$ in probability, and $S_3^2 \to \theta_3$ in probability; also, $S_1^2 - S_3^2 \to \theta_1 - \theta_3$ in probability.

Begin with a $1-\alpha$ exact confidence interval for θ_2 given by

$$\theta_2 \leq S_2^2 F_{\alpha:\infty,n_2}$$

Add $n_1 S_3^2$ to the right side and the "equivalent" value $n_1\theta_3$ to the left side to obtain

$$\theta_2 + n_1\theta_3 \leq S_2^2 F_{\alpha:\infty,n_2} + n_1 S_3^2$$

Take the inverse of each side, then multiply the right and left sides respectively, by $S_1^2 - S_3^2$ and its "equivalent" value $\theta_1 - \theta_3$

to obtain

$$\frac{S_1^2 - S_3^2}{S_2^2 F_{\alpha:\infty,n_2} + n_1 S_3^2} \le \frac{\theta_1 - \theta_3}{\theta_2 + n_1\theta_3}$$

Hence, when $I \to \infty, L(F_1,F_2)$ is required to behave like

$$L(F_1,F_2) = \frac{F_1 - 1}{F_2 F_{\alpha:\infty,n_2} + n_1}.$$

5) If an α-level test of $H_o:\theta_1 = \theta_3$ vs. $H_1:\theta_1 > \theta_3$ is accepted the confidence interval should include "zero" and $L(F_1,F_2)$ should be increasing in F_1 for fixed values of F_2.

H_o is accepted iff $F_1 \le F_{\alpha:n_1,n_3}$, so the confidence limit should be

$L(F_1,F_2) = 0$ when $F_1 \le F_{\alpha:n_1,n_3}$,

$L(F_1,F_2)$ monotonic increasing in F_1 for fixed values of F_2 when $F_1 > F_{\alpha:n_1,n_3}$.

The form of the function that could be used for $L(F_1,F_2)$ is

$$L(F_1,F_2) = \frac{a_o + a_1F_1}{b_o + b_1F_2}$$

but a more general function will be examined, i.e.,

$$L(F_1,F_2) = \frac{a_o + a_1F_1 + a_2F_1^{-1}}{b_o + b_1F_2 + b_2F_2^{-1}}$$

where a_i and b_i which are functions of n_1,n_2,n_3,α are determined by requiring $L(F_1,F_2)$ to satisfy conditions 1), 2), 3), 4), 5)

Substituting condition 1) gives

$$\frac{a_1}{b_1} = F^{-1}_{\alpha:n_1,n_2} \tag{2.2}$$

Substituting condition 2) gives

$$a_1 = F^{-1}_{\alpha:n_1,\infty} \tag{2.3}$$

and hence

$$b_1 = F^{-1}_{\alpha:n_1,\infty} F_{\alpha:n_1,n_2} \tag{2.4}$$

From condition 3) $\lim\limits_{J\to\infty} a_0 = -1$, $\lim\limits_{J\to\infty} b_0 = n_1$, $\lim\limits_{J\to\infty} a_1 = F^{-1}_{\alpha:n_1,\infty}$,

$\lim\limits_{J\to\infty} b_1 = 1$, $\lim\limits_{J\to\infty} a_2 = 0$, $\lim\limits_{J\to\infty} b_2 = 0$

which are satisfied for those values of a_1, b_1 from equations (2.3) and (2.4).

From condition 4) $\lim\limits_{I\to\infty} a_0 = -1$, $\lim\limits_{I\to\infty} b_0 = n_1$, $\lim\limits_{I\to\infty} a_1 = 1$,

$\lim\limits_{I\to\infty} b_1 = F_{\alpha:\infty,\ n_2}$, $\lim\limits_{I\to\infty} a_2 = 0$, $\lim\limits_{I\to\infty} b_2 = 0$.

Hence the limits of a_1 and b_1 are consistent for condtions 3) and 4).

From condition 5) $L(F_1,F_2) = 0$ if $F_1 \le F_{\alpha:n_1,n_3}$ which implies that

$$a_0 + a_1F_1 + a_2F_1^{-1} = 0 \quad \text{if} \quad F_1 \le F_{\alpha:n_1,n_3} .$$

Now substitute $F_1 = F_{\alpha:n_1,n_3}$ into

$$a_0 + a_1F_{\alpha:n_1,n_3} + a_2F^{-1}_{\alpha:n_1,n_3} = 0 ,$$

let a_1 be the value given in equation (2.3) and solve for a_2 to obtain

$$a_2 = (-a_0 - F^{-1}_{\alpha:n_1,\infty}F_{\alpha:n_1,n_3})F_{\alpha:n_1,n_3}$$

Use the simplest values for a_o and b_o, namely, $a_o = -1$ and $b_o = n_1$, and obtain

$$L(F_1,F_2) = \frac{-1 + F^{-1}_{\alpha:n_1,\infty} F_1 + (1-F^{-1}_{\alpha:n_1,\infty} F_{\alpha:n_1,n_3}) F_{\alpha:n_1,n_3} F_1^{-1}}{n_1 + F^{-1}_{\alpha:n_1,\infty} F_{\alpha:n_1,n_2} F_2 + b_2 F_2^{-1}}$$

$$\text{if } F_1 > F_{\alpha:n_1,n_3} \tag{2.5}$$

$L(F_1,F_2) = 0$ if $F_1 \le F_{\alpha:n_1,n_3}$

To determine the value of b_2 we impose condition 6).

Condition 6). When $H_o:\sigma_T^2 = 0$ vs. $H_1:\sigma_T^2 > 0$ is accepted the confidence interval is required to be exact.

To examine condition 7) we get $H_o:\sigma_T^2 = 0$ vs. $H_1:\sigma_T^2 > 0 \Longleftrightarrow$ $H_o:\theta_2 = \theta_3$ vs. $H_1:\theta_2 > \theta_3 \Longleftrightarrow H_o:\theta_A = (\theta_1-\theta_3)/I\theta_3$ vs. $H_1:\theta_A < (\theta_1-\theta_3)/I\theta_3$.

If H_o is true then $\theta_A = J\sigma_A^2/I\sigma_\varepsilon^2 = \frac{1}{I}\left(\frac{\theta_1}{\theta_3} - 1\right)$

So if H_o is accepted at α level we want $L(F_1,F_2)$ to be an exact $1-\alpha$ lower confidence point for θ_A .

H_o is accepted iff $F_2 \le F_{\alpha:n_2,n_3}$. An exact $1-\alpha$ lower confidence point for $\theta_1/I\theta_3 - 1/I$ is

$$\frac{S_1^2 F^{-1}_{\alpha:n_1,n_3}}{I\, S_3^2} - \frac{1}{I} \le \frac{\theta_1 - \theta_3}{I\,\theta_3}$$

So when $F_2 \le F_{\alpha:n_2,n_3}$ we want

$$L(F_1,F_2) = \frac{S_1^2 F^{-1}_{\alpha:n_1,n_3}}{IS_3^2} - \frac{1}{I} .$$

This is not consistent with the first five conditions, so we only require the denominator of (2.5) to be I when condition 6) is satisfied.

So set

$$n_1 + F^{-1}_{\alpha:n_1,\infty} F_{\alpha:n_1,n_2} F_{\alpha:n_2,n_3} + b_2 F^{-1}_{\alpha:n_2,n_3} = I$$

and then

$$b_2 = (1-F^{-1}_{\alpha:n_1,\infty} F_{\alpha:n_1,n_2} F_{\alpha:n_2,n_3}) F_{\alpha:n_2,n_3}$$

and this satisfies $\lim_{J\to\infty} b_2 = 0$.

The final result is

$$L(F_1,F_2) = \frac{-1 + F^{-1}_{\alpha:n_1,\infty} F_1 + (1-F^{-1}_{\alpha:n_1,\infty} F_{\alpha:n_1,n_3}) F_{\alpha:n_1,n_3} F_1^{-1}}{n_1 + F^{-1}_{\alpha:n_1,\infty} F_{\alpha:n_1,n_2} F_2 + (1-F^{-1}_{\alpha:n_1,\infty} F_{\alpha n_1,n_2} F_{\alpha:n_2,n_3}) F_{\alpha:n_2,n_3} F_2^{-1}}$$

$$\text{if } \begin{cases} S_1^2/S_3^2 \geq F_{\alpha:n_1,n_3} \\ \text{and} \\ S_2^2/S_3^2 \geq F_{\alpha:n_2,n_3} \end{cases}$$

$$L(F_1,F_2) = \frac{-1 + F^{-1}_{\alpha:n_1,\infty} F_1 + (1-F^{-1}_{\alpha:n_1,\infty} F_{\alpha:n_1,n_3}) F_{\alpha:n_1,n_3} F_1^{-1}}{I} \qquad (2.6)$$

$$\text{if } \begin{cases} S_1^2/S_3^2 \geq F_{\alpha:n_1,n_3} \\ \text{and} \\ S_2^2/S_3^2 < F_{\alpha:n_2,n_3} \end{cases}$$

$$L(F_1,F_2) = 0 \qquad \text{if } S_1^2/S_3^2 \leq F_{\alpha:n_1,n_3}$$

Instead of using $L(F_1,F_2)$ in equation (2.6), the simpler function in equation (2.7) will be used; which is obtained from equation (2.5) by setting $b_2 = 0$. Note that $\lim_{I\to\infty} b_2$ in equation (2.6) satisfies the requirement in condition 5). So the lower confidence point for θ_A is

$$L_A(F_1,F_2) = \frac{-1 + F^{-1}_{\alpha:n_1,\infty}F_1 + (1-F^{-1}_{\alpha:n_1,\infty}F_{\alpha:n_1,n_3})F_{\alpha:n_1,n_3}F_1^{-1}}{n_1 + F^{-1}_{\alpha:n_1,\infty}F_{\alpha:n_1,n_2}F_2}$$

$$\text{if } S_1^2/S_3^2 > F_{\alpha:n_1,n_3} \qquad (2.7)$$

$$L_A(F_1,F_2) = 0 \quad \text{if } S_1^2/S_3^2 \le F_{\alpha:n_1,n_3}$$

Note that the confidence interval in equation (2.7) covers the maximum likelihood estimator of θ_A, say $\hat{\theta}_A$. To demonstrate this, the M.L. estimator of θ_A, denoted by $\hat{\theta}_A$ is

$$\hat{\theta}_A = (F_1-1)/(F_2+n_1) \quad \text{if } F_1 > 1$$

$$\hat{\theta}_A = 0 \quad \text{if } F_1 \le 1$$

and the confidence interval is

$$\frac{-1 + F^{-1}_{\alpha:n_1,\infty}F_1 + (1-F^{-1}_{\alpha:n_1,\infty}F_{\alpha:n_1,n_3})F_{\alpha:n_1,n_3}F_1^{-1}}{n_1 + F^{-1}_{\alpha:n_1,\infty}F_{\alpha:n_1,n_2}F_2} \le \theta_A < \infty$$

$$\text{if } F_1 > F_{\alpha:n_1,n_3}$$

$$0 \le \theta_A < \infty \quad \text{if } F_1 \le F_{\alpha:n_1,n_3}$$

Now

$$-1 + F^{-1}_{\alpha:n_1,\infty}F_1 + (1-F^{-1}_{\alpha:n_1,\infty}F_{\alpha:n_1,n_3})F_{\alpha:n_1,n_3}F_1^{-1} < F_1 - 1 ,$$

$$F^{-1}_{\alpha:n_1,\infty} < 1 ,$$

and

$$(1-F^{-1}_{\alpha:n_1,\infty}F_{\alpha:n_1,n_2}) < 0.$$

Also

$$n_1 + F^{-1}_{\alpha:n_1,\infty}F_{\alpha:n_1,n_2}F_2 > F_2 + n_1$$

because

$$F^{-1}_{\alpha:n_1,\infty}F_{\alpha:n_1,n_2} > 1$$

So $L_A(F_1,F_2) < \hat{\theta}_A < \infty$ and $\hat{\theta}_A$ is in the $1-\alpha$ confidence interval.

Next consider a lower $1-\alpha$ confidence interval for θ_A. A lower $1-\alpha$ confidence interval for θ_A, given by $0 \leq \theta_A \leq U_A(F_1,F_2)$ can be readily obtained by replacing α by $1-\alpha$ in equation (2.7). The upper $1-\alpha$ confidence point of the lower confidence interval is given by

$$U_A(F_1,F_2) = \frac{-1 + F^{-1}_{1-\alpha:n_1,\infty}F_1 + (1-F^{-1}_{1-\alpha:n_1,\infty}F_{1-\alpha:n_1,n_3})F_{1-\alpha:n_1,n_3}F_1^{-1}}{n_1 + F^{-1}_{1-\alpha:n_1,\infty}F_{1-\alpha:n_1,n_2}F_2} \qquad \text{if } F_1 > F_{1-\alpha:n_1,n_3} \tag{2.8}$$

$$U_A(F_1,F_2) = 0 \qquad \text{if } F_1 \leq F_{1-\alpha:n_1,n_3}$$

The procedure for determining this lower confidence point is similar to that described for the upper confidence point.

From the formulas for upper and lower confidence intervals in equation (2.7) and (2.8), the two-sided confidence interval on θ_A

can be readily obtained. The $1-\alpha$ two-sided confidence interval for θ_A is given by $L_A^* \le \theta_A \le U_A^*$, where

$$L_A^* = \frac{-1 + F^{-1}_{\alpha/2:n_1,\infty}F_1 + (1-F_{\alpha/2:n_1,\infty}F_{\alpha/2:n_1,n_3})F_{\alpha/2:n_1,n_3}F_1^{-1}}{n_1 + F^{-1}_{\alpha/2:n_1,\infty}F_{\alpha/2:n_1,n_2}F_2}$$

$$U_A^* = \frac{-1 + F_{1-\alpha/2:n_1,\infty}F_1 + (1-F^{-1}_{1-\alpha/2:n_1,\infty}F_{1-\alpha/2:n_1,n_3})F_{1-\alpha/2:n_1,n_3}F_1^{-1}}{n_1 + F^{-1}_{1-\alpha/2:n_1,\infty}F_{1-\alpha/2:n_1,n_2}F_2} \tag{2.9}$$

This will be proved in Chapter 3.

If either limit is negative it is replaced by zero.

As mentioned in Chapter 1, due to the symmetry in the model, if a confidence interval for θ_A is obtained, the confidence interval for θ_T can be easily obtained. The formulas for the confidence intervals for θ_T are given below where $\theta_T = [\sigma_T^2/(\sigma_A^2+\sigma_\varepsilon^2)]$. The $1-\alpha$ lower confidence point for θ_T is

$$L_T(F_1,F_2) = \frac{-1 + F^{-1}_{\alpha:n_2,\infty}F_2 + (1-F^{-1}_{\alpha:n_2,\infty}F_{\alpha:n_2,n_3})F_{\alpha:n_2,n_3}F_2^{-1}}{n_2 + F^{-1}_{\alpha:n_2,\infty}F_{\alpha:n_2,n_1}F_1} \tag{2.10}$$

The $1-\alpha$ upper confidence point for θ_T is

$$U_T(F_1,F_2) = \frac{-1 + F^{-1}_{1-\alpha:n_2,\infty}F_2 + (1-F^{-1}_{1-\alpha:n_2,\infty}F_{1-\alpha:n_2,n_3})F_{1-\alpha:n_2,n_3}F_2^{-1}}{n_2 + F^{-1}_{1-\alpha:n_2,\infty}F_{1-\alpha:n_2,n_1}F_1} \tag{2.11}$$

The $1-\alpha$ two-sided confidence points for θ_T are

$$U_T = \frac{-1 + F^{-1}_{\alpha/2:n_2,\infty}F_2 + (1-F^{-1}_{\alpha/2:n_2,\infty}F_{\alpha/2:n_2,n_3})F_{\alpha/2:n_2,n_3}F_2^{-1}}{n_2 + F^{-1}_{\alpha/2:n_2,\infty}F_{\alpha/2:n_2,n_1}F_1}$$

and

$$L_T = \frac{-1 + F^{-1}_{1-\alpha/2:n_2,\infty}F_2 + (1-F^{-1}_{1-\alpha/2:n_2,\infty}F_{1-\alpha/2:n_2,n_3})F_{1-\alpha/2:n_2,n_3}F_2^{-1}}{n_2 + F^{-1}_{1-\alpha/2:n_2,\infty}F_{1-\alpha/2:n_2,n_1}F_1} \tag{2.12}$$

If any limit is negative it is replaced with zero.

From the confidence intervals obtained for θ_A and θ_T, and by equation (2.1) we get the corresponding confidence intervals for ρ_A and ρ_T.

The $1-\alpha$ lower confidence interval for ρ_A is

$$0 \le \rho_A \le \left(\frac{IL_A}{J + IL_A}\right)$$

where $L_A = L_A(F_1,F_2)$ which is defined in equation (2.7).

The $1-\alpha$ upper confidence intervals for ρ_A is

$$\left(\frac{IU_A}{J + IU_A}\right) \le \rho_A < \infty$$

where $U_A = U_A(F_1,F_2)$ which is defined in equation (2.8).

The $1-\alpha$ two-sided confidence interval for ρ_A is

$$\left(\frac{IU_A^*}{J + IU_A^*}\right) \le \rho_A \le \left(\frac{IL_A^*}{J + IL_A^*}\right)$$

where L_A^* and U_A^* are defined in equation (2.9).

The $1-\alpha$ lower confidence interval for ρ_T is

$$0 \leq \rho_T \leq \left(\frac{IL_T}{J + IL_T}\right)$$

where $L_T = L_T(F_1, F_2)$ which is defined in equation (2.10).

The $1-\alpha$ upper confidence interval for ρ_T is

$$\left(\frac{IU_T}{J + IU_T}\right) \leq \rho_T < \infty$$

where $U_T = U_T(F_1, F_2)$ which is defined in equation (2.11).

The $1-\alpha$ two-sided confidence interval for ρ_T is

$$\left(\frac{IU_T^*}{J + IU_T^*}\right) \leq \rho_T \leq \left(\frac{IL_T^*}{J + IL_T^*}\right)$$

Where U_T^* and L_T^* are the confidence points given in the equation (2.12).

3. Evaluation of the Methods

In the preceding section confidence point(s) were determined for upper, lower and two-sided confidence intervals which have exact an confidence coefficient $1-\alpha$ for specified restrictions, and for general conditions the confidence coefficient should be close to $1-\alpha$. In this section methods will be derived to determine how close the approximate confidence coefficients are to the specified ones.

First consider the upper confidence interval

$$P[L(F_1,F_2) \leq \theta_A < \infty] = P_L(\theta_A) \tag{3.1}$$

From equation (2.7)

$$L(F_1,F_2) = \begin{cases} \dfrac{-1+F^{-1}_{\alpha:n_1,\infty}F_1+(1-F^{-1}_{\alpha:n_1,\infty}F_{\alpha:n_1,n_3})F_{\alpha:n_1,n_3}F_1^{-1}}{n_1+F^{-1}_{\alpha:n_1,\infty}F_{\alpha:n_1,n_2}F_2} & \text{if } F_1 > F_{\alpha:n_1,n_3} \\ 0 & \text{if } F_1 \leq F_{\alpha:n_1,n_3} \end{cases} \tag{3.2}$$

From Section 1

$F_i = S_i^2/S_3^2$ for $i = 1,2$; also $n_iS_i^2/\theta_i = U_i$

for $i = 1,2,3$ are independent chi-square random variables with n_i degrees of freedom respectively.

Consider the following events

$$A = \{L(F_1,F_2) \leq \theta_A;\ \ F_1 > 0;\ \ F_2 > 0\}$$

$$B_1 = \{F_1 > F_{\alpha:n_1,n_3};\ \ F_2 > 0;\ \ L(F_1,F_2) \geq 0\}$$

$$B_2 = \{F_1 \le F_{\alpha:n_1,n_3};\ \ F_2 > 0;\ \ L(F_1,F_2) \ge 0\} \qquad (3.3)$$

So $B_1 \cap B_2 = \emptyset$ and $B_1 \cup B_2 = \Omega$ where

$$\Omega = \left\{\left(F_1,F_2,L(F_1,F_2)\right) : F_1 > 0;\ \ F_2 > 0;\ L(F_1,F_2) \ge 0\right\} .$$

Clearly $P_L(\theta_A) = P(A)$. We will write events B_1 and B_2 in (3.3) as

$$B_1 = \{F_1 > F_{\alpha:n_1,n_3}\}\ ;\ \ B_2 = \{F_1 \le F_{\alpha:n_1,n_3}\} \qquad (3.4)$$

From this we obtain

$$P_L(\theta_A) = P(A|B_1)P(B_1) + P(A|B_2)P(B_2)$$

$$P_L(\theta_A) = P[L(F_1,F_2) \le \theta_A | F_1 > F_{\alpha:n_1,n_3}]P[F_1 > F_{\alpha:n_1,n_3}]$$

$$+ P[L(F_1,F_2) \le \theta_A | F_1 \le F_{\alpha:n_1,n_3}]P[F_1 \le F_{\alpha:n_1,n_3}]$$

$$P_L(\theta_A) = P\left[\frac{-1+F^{-1}_{\alpha:n_1,\infty}F_1+(1-F^{-1}_{\alpha:n_1,\infty}F_{\alpha:n_1,n_3})F_{\alpha:n_1,n_3}F_1^{-1}}{n_1 + F^{-1}_{\alpha:n_1,\infty}F_{\alpha:n_1,n_2}F_2} \le \theta_A \middle| F_1 > F_{\alpha:n_1,n_3}\right]$$

$$\times P[F_1 > F_{\alpha:n_1,n_3}] + P[0 \le \theta_A | F_1 \le F_{\alpha:n_1,n_3}]P[F_1 \le F_{\alpha:n_1,n_3}]$$

But θ_A is always greater or equal to zero so

$$P[L(F_1,F_2) \le \theta_A \mid F_1 \le F_{\alpha:n_1,n_3}] = 1$$

and

$$P_L(\theta_A) = P\left[\frac{-1+F^{-1}_{\alpha:n_1,\infty}F_1+(1-F^{-1}_{\alpha:n_1,\infty}F_{\alpha:n_1,n_3})F_{\alpha:n_1,n_3}F_1^{-1}}{n_1 + F^{-1}_{\alpha:n_1,\infty}F_{\alpha:n_1,n_2}F_2} \le \theta_A \middle| F_1 > F_{\alpha:n_1,n_3}\right]$$

$$\times\ P[F_1 > F_{\alpha:n_1,n_3}] + P[F_1 \le F_{\alpha:n_1,n_3}] = P_1 + P_2 \qquad (3.5)$$

The first term in (3.5) gives

$$P_1 = P\left[\frac{-1+F^{-1}_{\alpha:n_1,\infty}F_1+(1-F^{-1}_{\alpha:n_1,\infty}F_{\alpha:n_1,n_3})F_{\alpha:n_1,n_3}F_1^{-1}}{n_1 + F^{-1}_{\alpha:n_1,\infty}F_{\alpha:n_1,n_2}F_2} \le \theta_A \middle| F_1 > F_{\alpha:n_1,n_3}\right]$$

$$= P\left[\frac{-F_{\alpha:n_1,\infty}+F_1+(F_{\alpha:n_1,\infty}-F_{\alpha:n_1,n_3})F_{\alpha:n_1,n_3}F_1^{-1}}{n_1F_{\alpha:n_1,\infty} + F_{\alpha:n_1,n_2}F_2} \le \theta_A \middle| F_1 > F_{\alpha:n_1,n_3}\right]$$

$$P_1 = P\left[\frac{-F_{\alpha:n_1,\infty}S_1^2S_3^2+S_1^4+(F_{\alpha:n_1,\infty}-F_{\alpha:n_1,n_3})F_{\alpha:n_1,n_3}S_3^4}{n_1F_{\alpha:n_1,\infty}S_3^2S_1^2 + F_{\alpha:n_1,n_2}S_2^2S_1^2} \le \theta_A \middle| F_1 > F_{\alpha:n_1,n_3}\right]$$

$$P_1 = P[-F_{\alpha:n_1,\infty}S_1^2S_3^2 + S_1^4 + (F_{\alpha:n_1,\infty}-F_{\alpha:n_1,n_3})F_{\alpha:n_1,n_3}S_3^4 \le$$

$$n_1F_{\alpha:n_1,\infty}S_1^2S_3^2\theta_A + F_{\alpha:n_1,n_2}S_2^2S_1^2\theta_A \mid F_1 > F_{\alpha:n_1,n_3}]$$

$$P_1 = P\left[\left(\frac{n_1^2S_1^4}{\theta_1^2}\right)\frac{\theta_1^2}{n_1^2} + (F_{\alpha:n_1,\infty}-F_{\alpha:n_1,n_3})F_{\alpha:n_1,n_3}\left(\frac{n_3^2S_3^4}{\theta_3^2}\right)\frac{\theta_3^2}{n_3^2} \le\right.$$

$$F_{\alpha:n_1,\infty}(1+n_1\theta_A)\left(\frac{n_1S_1^2}{\theta_1}\right)\left(\frac{n_3S_3^2}{\theta_3}\right)\frac{\theta_1\theta_3}{n_1n_3} + F_{\alpha:n_1,n_2}\theta_A\left(\frac{n_1S_1^2}{\theta_1}\right)$$

$$\left.\times\left(\frac{n_2S_2^2}{\theta_2}\right)\frac{\theta_1\theta_2}{n_1n_2}\;\middle|\; F_1 > F_{\alpha:n_1,n_3}\right]$$

$$P_1 = P\left[\frac{U_1^2\theta_1^2}{n_1^2} + (F_{\alpha:n_1,\infty}-F_{\alpha:n_1,n_3})F_{\alpha:n_1,n_3}\frac{U_3^2\theta_3^2}{n_3^2} \le F_{\alpha:n_1,\infty}(1+n_1\theta_A)\right.$$

$$\left.\times\, U_1U_3\frac{\theta_1\theta_3}{n_1n_3} + F_{\alpha:n_1,n_2}\theta_A\, U_1U_2\frac{\theta_1\theta_2}{n_1n_2}\;\middle|\; F_1 > F_{\alpha:n_1,n_3}\right]$$

where $U_i = n_iS_i^2/\theta_i$ for $i = 1,2,3$.

Divide both sides of the inequality by θ_3^2 and let $\rho_i = \theta_i/\theta_3$ for $i = 1,2$ to obtain

$$P_1 = P\left[\frac{U_1^2\rho_1^2}{n_1^2} + (F_{\alpha:n_1,\infty}-F_{\alpha:n_1,n_3})F_{\alpha:n_1,n_3}\frac{U_3^2}{n_3^2} \le F_{\alpha:n_1,\infty}(1+n_1\theta_A)\right.$$

$$\left.\times\frac{U_1U_3}{n_1n_3}\rho_1 + F_{\alpha:n_1,n_2}\,\theta_A\frac{\rho_1\rho_2}{n_1n_2}\,U_1U_2\;\middle|\; F_1 > F_{\alpha:n_1,n_3}\right]$$

Divide both sides of the inequality by U_1 to obtain

$$P_1 = P\left[\frac{\rho_1^2}{n_1^2}\,U_1 + (F_{\alpha:n_1,\infty}-F_{\alpha:n_1,n_3})\frac{F_{\alpha:n_1,n_3}}{n_3^2}\,\frac{U_3^2}{U_1} \le F_{\alpha:n_1,\infty}(1+n_1\theta_A)\right.$$

$$\left.\times\frac{U_3\rho_1}{n_1n_3} + F_{\alpha:n_1,n_2}\,\theta_A\frac{\rho_1\rho_2}{n_1n_2}\,U_2\;\middle|\; F_1 > F_{\alpha:n_1,n_3}\right]$$

$$P_1 = P\left[\left(\frac{\rho_1^2}{n_1^2} U_1 + (F_{\alpha:n_1,\infty} - F_{\alpha:n_1,n_3}) \frac{F_{\alpha:n_1,n_3}}{n_3^2} \frac{U_3^2}{U_1} - F_{\alpha:n_1,\infty}(1+n_1\theta_A)\right.\right.$$

$$\left.\left. \times \frac{\rho_1 U_3}{n_1 n_3}\right) \frac{n_1 n_2}{\theta_A \rho_1 \rho_2} \frac{1}{F_{\alpha:n_1,n_2}} \leq U_2 \middle| F_1 > F_{\alpha:n_1,n_3}\right]$$

$$P_1 = P\left[\frac{n_2\rho_1}{n_1\theta_A\rho_2 F_{\alpha:n_1,n_2}} U_1 + \frac{(F_{\alpha:n_1,\infty} - F_{\alpha:n_1,n_3})F_{\alpha:n_1,n_3}}{n_3\rho_1\rho_2\theta_A F_{\alpha:n_1,n_2}} \frac{U_3^2}{U_1}\right.$$

$$\left. - \frac{(1+n_1\theta_A)F_{\alpha:n_1,\infty}}{n_1\rho_2\theta_A F_{\alpha:n_1,n_2}} U_3 \leq U_2 \middle| F_1 > F_{\alpha:n_1,n_3}\right]$$

Let $C_1 = (n_2\rho_1)/(n_1\theta_A\rho_2 F_{\alpha:n_1,n_2})$

$$C_2 = (F_{\alpha:n_1,\infty} - F_{\alpha:n_1,n_3})F_{\alpha:n_1,n_3} \Big/ (n_3\rho_1\rho_2\theta_A F_{\alpha:n_1,n_2}) \qquad (3.6)$$

$$C_3 = (1+n_1\theta_A)F_{\alpha:n_1,\infty} \Big/ (n_1\rho_2\theta_A F_{\alpha:n_1,n_2})$$

Note $C_1 > 0$, $C_2 < 0$ and $C_3 > 0$. Then

$$P_1 = P[C_1U_1 + C_2 \frac{U_3^2}{U_1} - C_3U_3 \leq U_2 | F_1 > F_{\alpha:n_1,n_3}]$$

$$P_1 = P\left[\frac{C_1U_1(U_1+U_3)}{(U_1+U_3)} + \frac{C_2U_3^2}{(U_1+U_3)^2} \frac{(U_1+U_3)^2}{U_1} - \frac{C_3U_3}{(U_1+U_3)} (U_1+U_3) \leq \right.$$

$$\left. U_2 \middle| F_1 > F_{\alpha:n_1,n_3}\right]$$

Let $X = U_1 + U_3$, $Y = \frac{U_1}{U_1 + U_3}$, so $(1-Y) = \frac{U_3}{U_1 + U_3}$

Then

$$P_1 = P[C_1YX + C_2(1-Y)^2Y^{-1}X - C_3(1-Y)X \leq U_2 | F_1 > F_{\alpha:n_1,n_3}]$$

and

$$P_L(\theta_A) = P[C_1YX + C_2\,(1-Y)^2\,Y^{-1}X - C_3(1-Y)X \leq U_2;\ F_1 > F_{\alpha:n_1,n_3}]$$

$$+ P[F_1 \leq F_{\alpha:n_1,n_3}] \qquad (3.7)$$

The random variables X, Y, U_2 in (3.7) are independent and

$$X \sim \chi^2(n_1+n_2)\ ,\quad Y \sim B\left(\frac{n_1}{2}\ ,\ \frac{n_3}{2}\right)\ ,\quad U_2 \sim \chi^2\,(n_2)$$

Now consider the event $\{F_1 > F_{\alpha:n_1,n_3}\}$. The following events are equivalent:

$$\{F_1 > F_{\alpha:n_1,n_3}\};\ \{S_1{}^2 > S_3{}^2\,F_{\alpha:n_1,n_3}\};\ \left\{\frac{n_1S_1{}^2}{\theta_1}\ \frac{\theta_1}{n_1} > \frac{n_3S_3{}^2}{\theta_3}\ \frac{\theta_3}{n_3}\,F_{\alpha:n_1,n_3}\right\};$$

$$\left\{\frac{U_1\rho_1}{n_1} > \frac{U_3}{n_3}\,F_{\alpha:n_1,n_3}\right\};\ \left\{\frac{U_1}{U_3} > \frac{n_1}{n_3\rho_1}\,F_{\alpha:n_1,n_3}\right\};\ \left\{\frac{U_1}{U_3} + 1 > \frac{F_{\alpha:n_1,n_3}}{n_2\rho_1} + 1\right\};$$

$$\left\{(1-Y)^{-1} > \frac{F_{\alpha:n_1,n_3} + n_2\rho_1}{n_2\rho_1}\right\} \;;\; \left\{Y > \frac{F_{\alpha:n_1,n_3}}{F_{\alpha:n_1,n_3} + n_2\rho_1}\right\}. \qquad (3.8)$$

Then (3.7) can be written as

$$P_L(\theta_A) = \Big[\Big(C_1Y + C_2(1-Y)^2Y^{-1} - C_3(1-Y)\Big)X \leq U_2 \;;$$

$$Y > F_{\alpha:n_1,n_3}\Big/(F_{\alpha:n_1,n_3}+n_2\rho_1)\Big]$$

$$+ P\Big[Y \leq F_{\alpha:n_1,n_3}\Big/(F_{\alpha:n_1,n_3}+n_2\rho_1)\Big]$$

$$= P\left[\Big((C_1+C_2+C_3)Y+C_2Y^{-1} - (2C_2+C_3)\Big)X \leq U_2;\; Y > \lambda\right] + P[Y \leq \lambda]$$

$$= P[g(Y)X \leq U_2;\; Y > \lambda] + P[Y \leq \lambda] \qquad (3.9)$$

where

$$g(Y) = (C_1+C_2+C_3)Y + C_2Y^{-1} - (2C_2+C_3)$$

$$\lambda = F_{\alpha:n_1,n_3}\Big/(F_{\alpha:n_1,n_3} + n_2\rho_1) .$$

With probability one $Y \in (0,1]$.

Consider the function

$$g(y) = \frac{(C_1+C_2+C_3)y^2 - (2C_2+C_3)y + C_2}{y} \qquad 0 < y \leq 1$$

Clearly

$$g(y) = 0 \text{ iff } (C_1+C_2+C_3)y^2 - (2C_2+C_3)y + C_2 = 0;$$

hence, $g(y)$ has only two zeros for $y \in R$, and they are

$$\delta_1 = \frac{(2C_2+C_3) + [C_3^2-4C_1C_2]^{1/2}}{2(C_1+C_2+C_3)}$$

$$\delta_2 = \frac{(2C_2+C_3) - [C_3^2-4C_1C_2]^{1/2}}{2(C_1+C_2+C_3)}$$

Also $\lim_{y\to 0^+} g(y) = -\infty$ and $\lim_{y\to 1} g(y) = C_1 > 0$. Since $g(y)$ is continuous in $(0,1]$ then at least one of the roots (δ_1,δ_2) is in $(0,1)$. But

$$\frac{dg(y)}{dy} = (C_1+C_2+C_3) - C_2y^{-2} = C_1+C_3+C_2(1-y^{-2}) , \quad 0 < y \leq 1$$

But for $y \in (0,1)$, $\left(1 - \frac{1}{y^2}\right) < 0$. Also, $(C_1+C_3) > 0$ and $C_2 < 0$, hence, if $y \in (0,1)$ then

$$\frac{dg(y)}{dy} > 0 .$$

So $g(y)$ is monotonic increasing in $(0,1]$; also since

$\lim_{y\to 0^+} g(y) = -\infty$ and $\lim_{y\to 1} g(y) = C_1 > 0$, only one of the two roots can be in $(0,1]$. This gives the following two possible forms for $g(y)$.

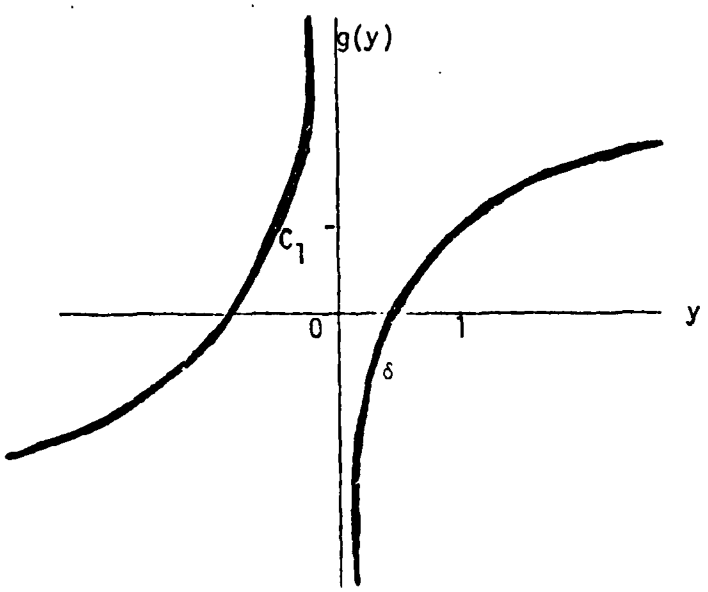

Figure 3.1

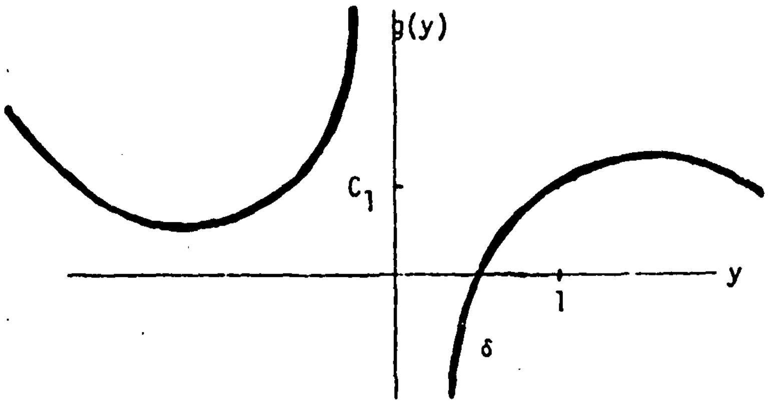

Figure 3.2

The number δ is the root in (0,1); also $\lambda \epsilon (0,1)$ so if $g(\lambda) < 0$ this implies that $\lambda < \delta$. Next we show that $g(\lambda) < 0$ for some value of $\lambda \epsilon (0,1)$.

$$g(\lambda) = (C_1+C_2+C_3)\lambda + C_2\lambda^{-1} - (2C_2+C_3)$$

$$= \frac{(C_1+C_2+C_3)F_{\alpha:n_1,n_3}}{(F_{\alpha:n_1,n_3} + n_2\rho_1)} + C_2 \frac{(F_{\alpha:n_1,n_3}+n_2\rho_1)}{F_{\alpha:n_1,n_3}} - (2C_2+C_3)$$

$$= \frac{(C_1+C_2+C_3)F^2_{\alpha:n_1,n_3}+C_2(F_{\alpha:n_1,n_3}+n_2\rho_1)^2-(2C_2+C_3)F_{\alpha:n_1,n_3}(F_{\alpha:n_1,n_3}+n_2\rho_1)}{F_{\alpha:n_1,n_3}(F_{\alpha:n_1,n_3} + n_2\rho_1)}$$

First note that the denominator of $g(\lambda)$ is always positive. Denote the numerator by N so

$$N = (C_1+C_2+C_3)F^2_{\alpha:n_1,n_3} + C_2F^2_{\alpha:n_1,n_3} + C_2n_2^2\rho_1^2 + 2C_2n_2\rho_1F_{\alpha:n_1,n_3}$$

$$- 2C_2F^2_{\alpha:n_1,n_3} - 2C_2n_2\rho_1F_{\alpha:n_1,n_3} - C_3F^2_{\alpha:n_1,n_3} - C_3n_2\rho_1F_{\alpha:n_1,n_3}$$

$$= C_1F^2_{\alpha:n_1,n_3} + C_2n_2^2\rho_1^2 - C_3n_2\rho_1F_{\alpha:n_1,n_3}$$

Replace C_1, C_2, and C_3 by their values in (3.6) and obtain

$$N = \frac{n_2\rho_1 F^2_{\alpha:n_1,n_3}}{n_1\rho_2\theta_A F_{\alpha:n_1,n_2}} + \frac{n_2^2\rho_1^2 F_{\alpha:n_1,\infty} F_{\alpha:n_1,n_3} - n_2^2\rho_1^2 F^2_{\alpha:n_1,n_3}}{n_3\rho_1\rho_2\theta_A F_{\alpha:n_1,n_2}}$$

$$- \frac{n_2^2(\theta_A n_1+1)\rho_1 F_{\alpha:n_1,\infty} F_{\alpha:n_1,n_3}}{n_3\rho_2\theta_A F_{\alpha:n_1,n_2}}$$

$$N = \frac{\left[n_2^2\rho_1^2 F_{\alpha:n_1,n_3} - n_2^2\rho_1^2 F_{\alpha:n_1,n_3} + F_{\alpha:n_1,\infty} n_2^2\rho_1^2 - n_2^2\rho_1^2(\theta_A n_1+1)F_{\alpha:n_1,\infty}\right] F_{\alpha:n_1,n_3}}{n_3\rho_1\rho_2\theta_A F_{\alpha:n_1,n_2}}$$

$$= \frac{F_{\alpha:n_1,\infty} F_{\alpha:n_1,n_3} n_2^2\rho_1^2(1-\theta_A n_1+1)}{n_3\rho_1\rho_2\theta_A F_{\alpha:n_1,n_2}}$$

$$= \frac{-\theta_A n_1 n_2^2\rho_1^2 F_{\alpha:n_1,\infty} F_{\alpha:n_1,n_3}}{n_3\rho_1\rho_2\theta_A F_{\alpha:n_1,n_2}} < 0$$

So the numerator is always negative which implies that $g(\lambda)$ will always be negative; and hence, $\lambda < \delta$.

Summarizing these results give

a) $g(y) \geq 0$ iff $y \geq \delta$ for $y \in (0,1)$

b) $0 < \lambda < \delta < 1$ (3.10)

c) $g(y) < 0$ iff $y \in (\lambda,\delta)$

Using these facts to evaluate

$$P[g(Y)X \leq U_2;\ Y > \lambda] \text{ in } (3.9)$$

we obtain

$$= P[g(Y)X \le U_2;\ Y > \lambda | \lambda < Y \le \delta]\ \ P[\lambda < Y \le \delta]$$

$$+ P[g(Y)X \le U_2;\ Y > \lambda | Y > \delta]\ P[Y > \delta]$$

$$= 1 \cdot P[\lambda \le Y \le \delta] + [g(Y)X \le U_2;\ Y > \lambda;\ Y > \delta]$$

$$= P[\lambda \le Y \le \delta] + P[g(Y)X \le U_2;\ Y > \delta]$$

$$= F_Y(\delta) - F_Y(\lambda) + P\ [g(Y)X \le U_2;\ Y > \delta]$$

where

$$F_Y(y) = P[Y \le y]$$

From the original probability in (3.9) we get

$$P_L(\theta_A) = F_Y(\delta) - F_Y(\lambda) + P[g(Y)X \le U_2;\ Y > \delta] + F_Y\ (\lambda)$$

$$= F_Y(\delta) + P[g(Y)X \le U_2;\ Y > \delta] \tag{3.11}$$

Also

$$P[g(Y)X \le U_2;\ Y > \delta] = P[g(Y)X + X \le U_2 + X;\ Y > \delta]$$

$$= P\left[g(Y) + 1 \le \frac{U_2 + X}{X}\ ;\ Y > \delta\right]$$

$$= P\left[[g(Y) + 1]^{-1} \ge \frac{X}{X + U_2};\ Y > \delta\right] \tag{3.12}$$

Let $Z = X/(X+U_2)$; It follows that Z is distributed as Beta with parameters $\left(\frac{n_1+n_3}{2}\ ,\ \frac{n_2}{2}\right)$. Also, let n_i be an even integer so that $n_i/2 = m_i$ for $i = 1,2,3$ and hence the m_i are positive integers.

If X_1, X_2, X_3 are independent chi-square random variables with m_1, m_2, m_3 degrees of freedom, respectively, and

$$Y_1 = \frac{X_1}{X_1 + X_2}\ , \qquad Y_2 = \frac{X_1 + X_2}{X_1 + X_2 + X_3}\ ,$$

then Y_1, Y_2 are mutually independent Beta random variables with parameters $(m_1/2,\ m_2/2)$ and $\big((m_1+m_2)/2, m_3/2\big)$ respectively.

From the result in (3.12) we obtain (let v denote $[g(y)+1]^{-1}$)

$$P_L(\theta_A) = P\Big[[g(Y)+1]^{-1} \geq Z;\ Y > \delta\Big] + F_Y(\delta)$$

$$= F_Y(\delta) + \int_\delta^1\int_0^v \frac{B^{-1}(m_1,m_3)}{B(m_1+m_3,m_2)}\ y^{m_1-1}\ (1-y)^{m_3-1}\ z^{m_1+m_3-1}(1-z)^{m_2-1}\ dz\ dy$$

$$= F_Y(\delta) + \int_\delta^1\int_0^v \frac{B^{-1}(m_1,m_3)}{B(m_1+m_3,m_2)}\ y^{m_1-1}\ (1-y)^{m_3-1}\ z^{m_1+m_3-1}$$

$$\times \sum_{r=0}^{m_2-1} \binom{m_2-1}{r}(1)^r\ (-z)^{m_2-r-1}\ dz\ dy$$

$$= F_Y(\delta) + \sum_{r=0}^{m_2-1} \frac{B^{-1}(m_1,m_3)\Gamma(m_2)(-1)^{m_2-r-1}}{B(m_1+m_3,m_2)\Gamma(r+1)\Gamma(m_2-r)}$$

$$\times \int_\delta^1 y^{m_1-1}\ (1-y)^{m_3-1}\int_0^{[g(y)+1]^{-1}} z^{(m_1+m_2+m_3-r-2)}\ dz\ dy$$

$$P_L(\theta_A) = F_Y(\delta) + \sum_{r=0}^{m_2-1} \frac{B^{-1}(m_1,m_3)\Gamma(m_2)(-1)^{m_2-r-1}}{B(m_1+m_3,m_2)\Gamma(r+1)\Gamma(m_2-r)}$$

$$\times \int_{\delta}^{1} y^{m_1-1}(1-y)^{m_3-1} \frac{[g(y)+1]^{-(m_1+m_2+m_3-r-1)}}{(m_1+m_2+m_3-r-1)} dy$$

$$= F_Y(\delta) + \sum_{r=0}^{m_2-1} \frac{(-1)^{m_2-r-1} (m_1+m_2+m_3-1)!}{(m_1-1)!(m_3-1)!(m_2-r-1)!r!(m_1+m_2+m_3-r-1)}$$

$$\times \int_{\delta}^{1} \frac{y^{m_1-1}(1-y)^{m_3-1}}{[g(y)+1]^{(m_1+m_2+m_3-r-1)}} dy \qquad (3.13)$$

From the expression in (3.13) we will compute $P_L(\theta_A)$.

Next consider the lower confidence interval

$$P[\theta_A \leq U(F_1,F_2)] = P_U(\theta_A) \qquad (3.14)$$

From equation (2.8) it follows that

$$U(F_1,F_2) = \frac{-1 + F^{-1}_{1-\alpha:n_1,\infty}F_1 + (1-F^{-1}_{1-\alpha:n_1,\infty}F_{1-\alpha:n_1,n_3})F_{1-\alpha:n_1,n_3}F_1^{-1}}{n_1 + F^{-1}_{1-\alpha:n_1,\infty}F_{1-\alpha:n_1,n_2}F_2}$$

$$\text{if } S_1^2/S_3^2 > F_{1-\alpha:n_1,n_3}$$

$$U(F_1,F_2) = 0 \qquad \text{if } S_1^2/S_3^2 \leq F_{1-\alpha:n_1,n_3} \qquad (3.15)$$

Substitute equation (3.15) into (3.14) to obtain

$$P_U(\theta_A) = P\left[[g(Y) + 1]^{-1} \leq Z;\ Y \geq \delta\right] \qquad (3.16)$$

where $Y \sim \text{Beta}(m_1,m_3)$, and $Z \sim \text{Beta}(m_1+m_3,m_2)$; Y and Z are independent.

$$g(y) = (C_1+C_2+C_3)y + C_2y^{-1} - (C_3+2C_2) \tag{3.17}$$

δ is the largest root of $g(y)$.

The C's in $g(y)$ are obtained by changing the α's to $1-\alpha$'s in the equation (3.6). So we have

$$C_1 = \frac{n_2\rho_1}{n_1\theta_A\rho_2F_{1-\alpha:n_1,n_2}}$$

$$C_2 = \frac{(F_{1-\alpha:n_1,\infty}-F_{1-\alpha:n_1,n_3})F_{1-\alpha:n_1,n_3}}{n_3\rho_1\rho_2\theta_AF_{1-\alpha:n_1,n_2}} \tag{3.18}$$

$$C_3 = \frac{(1+n_1\theta_A)F_{1-\alpha:n_1,\infty}}{n_1\rho_2\theta_AF_{1-\alpha:n_1,n_2}}$$

Note $C_1 > 0$, $C_3 > 0$; and $C_2 > 0$ if $n_1 > 2$, $n_2 \geq 2$, and $\alpha \leq .10$.

Consider the function $g(y)$ in equation (3.17) . Clearly

$$g(y) = 0 \quad \text{iff} \quad (C_1+C_2+C_3)y - (2C_2+C_3)y + C_2 = 0;$$

hence, $g(y)$ has only two zeros for $y \in \mathbb{R}$ and they are

$$\delta_1 = \frac{(2C_2+C_3) + [C_3^{\,2}-4C_1C_2]^{1/2}}{2(C_1 + C_2 + C_3)}$$

$$\delta_2 = \frac{(2C_2+C_3) - [C_3^{\,2}-4C_1C_2]^{1/2}}{2(C_1 + C_2 + C_3)}$$

Also $\lim_{y\to 0^+} g(y) = \infty$ and $\lim_{y\to 1} g(y) = C_1 > 0$.

Since $g(y)$ is continuous in $(0,1]$, then either both of the two roots of $g(y)$ are in $(0,1]$, or neither of the roots are in the interval $(0,1]$, or the two roots are complex and not real. But $\lambda \in (0,1]$ so that $g(\lambda) < 0$. For

$$\lambda = \frac{F_{1-\alpha:n_1,n_3}}{F_{1-\alpha:n_1,n_3} + \rho_1 n_2}$$

$$g(\lambda) = \frac{-\theta_A \rho_1^2 n_1 n_2^2 F_{1-\alpha:n_1,\infty} F_{1-\alpha:n_1,n_3}}{n_3 \rho_1 \rho_2 \theta_A F_{1-\alpha:n_1,n_2}} < 0 \tag{3.19}$$

This follows from the proof of (b) in Equation (3.10).

From the above it is clear that the two roots of $g(y)$ are in $(0,1]$, which results in the following form for $g(y)$.

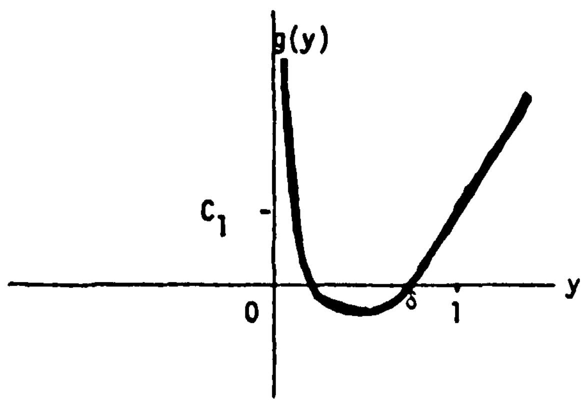

Figure 3.3

From equation (3.16) we evaluate $P_U(\theta_A)$ as follows

$$P_U(\theta_A) = \sum_{r=0}^{m_2-1} \frac{(-1)^{m_2-r-1}(m_1+m_2+m_3-1)!}{(m_3-1)!(m_1-1)!(m_2-r-1)!r!(m_1+m_2+m_3-r-1)}$$

$$\times \int_{\delta}^{1} y^{m_1-1} (1-y)^{m_3-1} \left[\frac{-1}{[g(y)+1]^{(m_1+m_2+m_3-r-1)}} + 1 \right] dy \tag{3.20}$$

The <u>two-sided</u> confidence interval is

$$P[L_1(F_1,F_2) \leq \theta_A \leq U_1(F_1,F_2)] = P_2(\theta_A) \tag{3.21}$$

where $L_1(F_1,F_2)$ is given by (3.2) with α replaced by $\alpha/2$ and $U_1(F_1,F_2)$ given by (3.15) with $1-\alpha$ replaced by $1-\alpha/2$.

To show (3.21) let A and B be the following events

$$A = \{L_1(F_1,F_2) \leq \theta_A\} \qquad B = \{\theta_A \leq U_1(F_1,F_2)\}$$

Then

$$P_2(\theta_A) = P(A \cap B) \tag{3.22}$$

From (3.22) we obtain

$$\begin{aligned} P_2(\theta_A) &= P(A \cap B) = 1 - P(\overline{A \cap B}) = 1 - P(\bar{A} \cup \bar{B}) \\ &= 1 - [P(\bar{A}) + P(\bar{B}) - P(\bar{A} \cap \bar{B})] \\ &= 1 - P(\bar{A}) - P(\bar{B}) + P(\bar{A} \cap \bar{B}) \end{aligned} \tag{3.23}$$

Also

$$P(\bar{A}) = 1 - P_L(\theta_A), \quad P(\bar{B}) = 1 - P_U(\theta_A)$$

and

$$P(\bar{A} \cap \bar{B}) = P\left(\theta_A \le L(F_1,F_2);\ \ U(F_1,F_2) \le \theta_A\right) = 0$$

if $L(F_1,F_2) \le U(F_1,F_2)$ with probability one. Next we will show that this is indeed the case for $n_1 > 2$, $n_2 \ge 2$ and $\alpha \le .10$. We must prove

$$\left\{\frac{-1 + F^{-1}_{\alpha:n_1,\infty}F_1 + (1-F^{-1}_{\alpha:n_1,\infty}F_{\alpha:n_1,n_3})\,F_{\alpha:n_1,n_3}F_1^{-1}}{n_1 + F^{-1}_{\alpha:n_1,\infty}F_{\alpha:n_1,n_2}F_2}\right.$$

$$\left.\le \frac{-1 + F^{-1}_{1-\alpha:n_1,\infty}F_1 + (1-F^{-1}_{1-\alpha:n_1,\infty}F_{1-\alpha:n_1,n_3})F_{1-\alpha:n_1,n_3}F_1^{-1}}{n_1 + F^{-1}_{1-\alpha:n_1,\infty}F_{1-\alpha:n_1,n_2}F_2}\right\}$$

which we write as

$$= \left\{\frac{-1 + A_1F_1 + B_1F_1^{-1}}{n_1 + C_1F_2} \le \frac{-1 + A_2F_1 + B_2F_1^{-1}}{n_1 + C_2F_2}\right\} \tag{3.24}$$

This implies

$$U(F_1,F_2) \le L(F_1,F_2)$$

To prove equation (3.24) we use Lemma 1, 2, and 3. (See Ghosh (1973)).

<u>Lemma 1</u>. $F_{\alpha:n_1,n_2}$ is decreasing in n_2 if $n_1 > 2$, $n_2 \ge 2$, $\alpha \le .10$.

<u>Lemma 2</u>. $F_{\alpha:n_1,n_2}$ is decreasing in n_1 if $n_1 > 2$, $n_2 \ge 2$, $\alpha \le .10$.

Lemma 3. $F_{1-\alpha:n_1,n_2}$ is increasing in n_2 if $n_1 > 2$, $n_2 \geq 2$, $\alpha \leq .10$.

Proof:

$F_{\alpha:n_1^*,n_2} \leq F_{\alpha:n_1,n_2}$ if $n_1^* > n_1$ (by Lemma 2).

So

$$F^{-1}_{1-\alpha:n_2,n_1^*} \leq F^{-1}_{1-\alpha\cdot n_2,n_1}$$

And from this the result follows.

Using the Lemmas in expression (3.24) we have

1) $C_2 \leq C_1$ since

$$F_{1-\alpha:n_1,\infty}F_{1-\alpha:n_1,n_2} \leq F^{-1}_{\alpha:n_1,\infty}F_{\alpha:n_1,n_2}$$

then

$$C_1 \geq 1 \ , \ C_2 \leq 1 \quad \text{so} \quad \frac{-1 + A_1F_1 + B_1F_1^{-1}}{n_1 + C_1F_2} \leq \frac{-1 + A_1F_1 + B_1F_1^{-1}}{n_1 + C_2F_2}$$

2) $F^{-1}_{\alpha:n_1,} \leq F^{-1}_{1-\alpha:n_1,}$ so $A_1 \leq A_2$

and

$$\frac{-1 + A_1F_1 + B_1F_1^{-1}}{n_1 + C_1F_2} \leq \frac{-1 + A_2F_1 + B_1F_1^{-1}}{n_2 + C_2F_2}$$

3) $F^{-1}_{\alpha:n_1,\infty}F_{\alpha:n_1,n_3} \geq F^{-1}_{1-\alpha:n_1,\infty}F_{1-\alpha:n_1,n_3}$

so

$$-F^{-1}_{\alpha:n_1,\infty}F_{\alpha:n_1,n_3} \leq -F^{-1}_{1-\alpha:n_1,\infty}F_{1-\alpha:n_1,n_3}$$

and

$$(1-F_{\alpha:n_1,\infty}F_{\alpha:n_1,n_3}) \leq (1-F^{-1}_{1-\alpha:n_1,\infty}F_{1-\alpha:n_1,n_3})$$

$$(1-F_{\alpha:n_1,\infty}F_{\alpha:n_1,n_3})F_{\alpha:n_1,n_3} \leq (1-F^{-1}_{1-\alpha:n_1,\infty}F_{1-\alpha:n_1,n_3})F_{1-\alpha:n_1,n_3}$$

then $$B_1 \leq B_2$$

and

$$\frac{-1 + A_1F_1 + B_1F_1^{-1}}{n_1 + C_1F_2} \leq \frac{-1 + A_2F_1 + B_2F_1^{-1}}{n_1 + C_2F_2}$$

Going back to (3.23) we get

$$P_2(\theta_A) = 1 - \left(1-P_L(\theta_A)\right) - \left(1-P_U(\theta_A)\right) + 0$$

$$P_2(\theta_A) = P_L(\theta_A) + P_U(\theta_A) - 1 \qquad (3.25)$$

Numerical integration will be used to evaluate the integrals in $P_L(\theta_A)$, $P_U(\theta_A)$, and $P_2(\theta_A)$. The method used is Simpson's Rule which gives

$$\int_a^b f(y)dy \simeq \frac{h}{3}\Big[f(a) + 4f(a+h) + 2f(a+2h) + 4f(a+3h) + 2f(a+4h) + \ldots + 2f\big(a+(n-2)h\big) + 4f\big(a+(n-1)h\big) + f(b)\Big]$$

where h is the length of the subintervals in which (a,b) has been partitioned. We used $h = .001$.

Various combinations of sample sizes were chosen, and for each of the combinations several sets of ρ_1 and ρ_2, and different α's were used.

A list of the values of I, J, ρ_1, ρ_2, and α used is given below

$$(I,J) : (3,3);(3,7);(3,11);(5,3);(5,5);(5,11);(7,3);$$
$$(7,7);(7,11);(9,3);(9,9);(9,11);(11,3);(11,7);(11,11)$$
$$\sigma_A^2/\sigma_\varepsilon^2 : .003,\ 1,\ 10,\ 100 \tag{4.1}$$
$$\sigma_T^2/\sigma_\varepsilon^2 : 0,\ 1,\ 10,\ 100$$
$$1-\alpha : .90,\ .95,\ .975,\ .99$$
$$\rho_1 = 1 + J\ \sigma_A^2/\sigma_\varepsilon^2$$
$$\rho_2 = 1 + I\ \sigma_T^2/\sigma_\varepsilon^2$$

The results are given in Tables 2, 3, and 4. The first five columns in each table show the I, J, n_1, n_2, and n_3 values. Each of the remaining columns give the results for the specified $1-\alpha$ confidence coefficients. For each of these columns the ranges that the confidence coefficients vary (as ρ_1 and ρ_2 vary over their allowable values) are shown.

The approximate confidence coefficients corresponding to the lower confidence intervals are very close to the specified ones, shown in Table 2. The approximate confidence coefficients for the upper confidence intervals range from values very close to the specified ones to some high values, shown in Table 3. Similar results hold for the two-sided confidence limits shown in Table 4.

For I fixed the range of the confidence coefficients is closer to the nominal value when J increases. For J fixed the upper bound of the range is higher when I increases. So the confidence coefficient will be better when J is large. However, in general the approximate confidence procedures seem adequate if I and J are each 5 or bigger.

Table 2

Ranges of confidence coefficients for lower confidence intervals on ρ_A .

I	J	n_1	n_2	n_3	$1 - \alpha = .95$	$1 - \alpha = .975$	$1 - \alpha = .99$
3	3	2	2	4	.94948 - .95010	.97457 - .97501	.98614 - .98999
3	7	2	6	12	.94974 - .95009	.97477 - .97524	.97696 - .98982
3	11	2	10	20	.94989 - .95103	.97364 - .97608	.98747 - .99115
5	3	4	2	8	.94769 - .95304	.97359 - .97672	.98829 - .99093
5	5	4	4	16	.94914 - .95184	.97451 - .97617	.98700 - .99102
5	11	4	10	40	.94978 - .95095	.97486 - .97560	.98225 - .99445
7	3	6	2	12	.94515 - .95560	.97187 - .97855	.98818 - .99172
7	7	6	6	36	.94901 - .95325	.97438 - .97703	.98669 - .99171
7	11	6	10	60	.94959 - .95230	.97472 - .97643	.98457 - .99297
9	3	8	2	16	.94293 - .95764	.97034 - .97979	.98743 - .99231
9	9	8	8	64	.94901 - .95421	.97436 - .97764	.98606 - .99260
9	11	8	10	80	.94930 - .95365	.97454 - .97729	.98514 - .99255
11	3	10	2	20	.94116 - .95959	.96912 - .98099	.98664 - .99287
11	7	10	6	60	.94798 - .95656	.97371 - .97914	.98691 - .99242
11	11	10	10	100	.94910 - .95497	.97442 - .97813	.98525 - .99334

Table 3

Ranges of confidence coefficients for upper confidence intervals on ρ_A .

I	J	n_1	n_2	n_3	$1 - \alpha = .95$	$1 - \alpha = .975$	$1 - \alpha = .99$
3	3	2	2	4	.95104 - .98322	.97572 - .99507	.99005 - .99911
3	7	2	6	12	.95032 - .96349	.97531 - .98467	.99007 - .99621
3	11	2	10	20	.94916 - .95811	.97464 - .98098	.98999 - .99353
5	3	4	2	8	.95079 - .98625	.97551 - .99601	.99005 - .99927
5	5	4	4	16	.95067 - .97506	.97545 - .99114	.99015 - .99812
5	11	4	10	40	.94888 - .96187	.97442 - .98334	.98986 - .99464
7	3	6	2	12	.95066 - .98772	.97542 - .99645	.99005 - .99934
7	7	6	6	36	.94938 - .97183	.97518 - .98926	.99009 - .99738
7	11	6	10	60	.94880 - .96474	.97436 - .98507	.98979 - .99540
9	3	8	2	16	.95057 - .98854	.97505 - .99670	.99005 - .99938
9	9	8	8	64	.94880 - .97006	.97426 - .98818	.98994 - .99671
9	11	8	10	80	.94876 - .96702	.97433 - .98639	.98959 - .99594
11	3	10	2	20	.95051 - .98904	.97506 - .99683	.99005 - .99940
11	7	10	6	60	.94906 - .97613	.97485 - .99138	.99020 - .99789
11	11	10	10	100	.94872 - .96889	.97431 - .98744	.98930 - .99673

Table 4

Range of confidence coefficients for a two-sided confidence interval on ρ_A

I	J	n_1	n_2	n_3	$1 - \alpha = .90$	$1 - \alpha = .95$
3	3	2	2	4	.90103 - .93208	.95072 - .97000
3	7	2	6	12	.90043 - .91349	.95030 - .96000
3	11	2	10	20	.89950 - .90849	.94827 - .95702
5	3	4	2	8	.90096 - .93896	.95062 - .97273
5	5	4	4	16	.90077 - .92684	.95051 - .96725
5	11	4	10	40	.89929 - .91272	.95031 - .95884
7	3	6	2	12	.90094 - .94310	.95604 - .97436
7	7	6	6	36	.89871 - .92475	.94984 - .96607
7	11	6	10	60	.89896 - .91673	.94955 - .96130
9	3	8	2	16	.90092 - .94619	.95059 - .97647
9	9	8	8	64	.89835 - .92374	.94929 - .96546
9	11	8	10	80	.89878 - .92016	.94938 - .96334
11	3	10	2	20	.90090 - .94863	.95058 - .97782
11	7	10	6	60	.89790 - .93203	.94926 - .97006
11	11	10	10	100	.89868 - .92315	.94927 - .96510

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